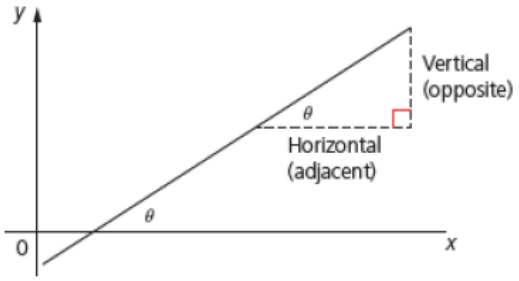
Higher Straight Line Work for week beginning Monday 4th May

Monday 4th May - Higher Straight Line: m = tan Θ

Gradient: $m = \tan\theta$

When you first studied gradient, you used the formula $m = \dfrac{\text{vertical height}}{\text{horizontal distance}}$



By looking at the triangle, you can see that the vertical side is *opposite* to the angle θ (theta) and the horizontal side is *adjacent* to the angle θ. So

$$m = \frac{\text{vertical height}}{\text{horizontal distance}} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta$$

The formula $m = \tan\theta$ tells us that the **gradient** of a straight line is the **tangent** of the angle that the line makes with the positive direction of the x-axis.

- A line making an angle of 45° with the positive *x*-axis has gradient 1 as $\tan 45° = 1$.
- A line with gradient 2 makes an angle of $\tan^{-1} 2 = 63{\cdot}4°$ with the positive *x*-axis.
- A line with zero gradient is parallel to the *x*-axis.
- A line with an undefined gradient is perpendicular to the positive *x*-axis.
- A line with a negative gradient makes an obtuse angle with the positive *x*-axis. For example, a line with gradient –2 makes an angle of $\tan^{-1}(-2) = 116{\cdot}6°$ with the positive *x*-axis.

Example

What angle does the line joining points A (–3, 5) and B (2, –1) make with the positive direction of the *x*-axis?

SOLUTION

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-1-5}{2-(-3)} = -\frac{6}{5} \Rightarrow \tan\theta = -\frac{6}{5} \Rightarrow \theta = 129{\cdot}8$$

Hence the line makes an angle of 129·8° with the positive direction of the *x*-axis.

Exercise 1.3

(Unless otherwise stated give angles correct to one decimal place and gradients correct to two decimal places.)

1 Find the size of the angle between the positive *x*-axis and a line with gradient:

a) 4 b) $\frac{1}{\sqrt{3}}$ c) –1

d) 0·5 e) $-\sqrt{3}$ f) 57

2 Find the gradient of a line inclined to the positive *x*-axis at an angle of:

a) 20° b) 120° c) 55°

d) 90° e) 78·7° f) 164°

3 a) Calculate the gradient of the line joining points A (5, 3) and B (8, 4). HPQ

b) What angle does this line make with the positive *x*-axis?

4 a) Calculate the gradient of the line joining points A (–3, 3) and B (4, –2)

b) What angle does this line make with the positive *x*-axis?

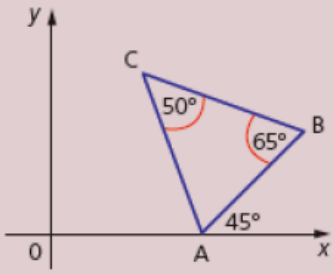
5 What angle does the line joining points P (2, 1) and Q (3, 7) make with the positive direction of the *x*-axis?

6 What angle does the line joining points R (–3, –3) and B (–7, –7) make with the positive direction of the *x*-axis?

7 What angle does the line joining points T (–1, 4) and U (3, 8) make with the positive direction of the *x*-axis?

8 Calculate the gradient, correct to two decimal places, of:

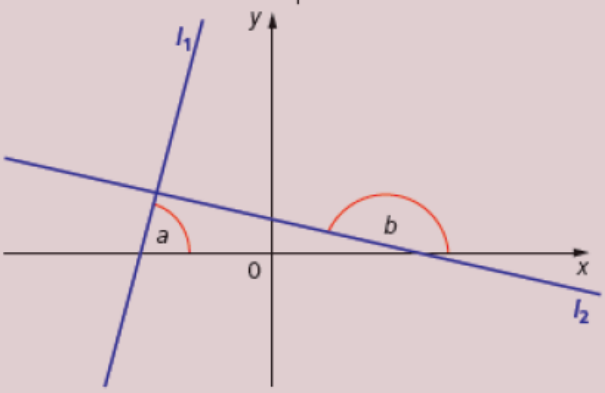
a) AB
b) AC
c) BC



9 The acute angle between two intersecting lines is 50°. One of the lines has gradient 3. Make a sketch and find the possible gradients of other line. Give your answers correct to one decimal place.

10 Line l_1 has gradient 4; line l_2 has gradient $-\frac{1}{4}$.

a) Calculate the size of the angles marked a and b.
b) Calculate the angle between lines l_1 and l_2.



11 **a)** Repeat question 10 for lines with gradients 3 and $-\frac{1}{3}$.
b) Comment on your findings.

Exercise 1.3

1 a) 76·0°
b) 30·0°
c) 135·0°
d) 26·6°
e) 120·0°
f) 89·0°

2 a) 0·36
b) −1·73
c) 1·43
d) Undefined
e) 5·00
f) −0·29

3 a) $\frac{1}{3}$
b) 18·4°

4 a) $-\frac{5}{7}$
b) 144·5°

5 80·5°

6 45·0°

7 45·0°

8 a) 1·00
b) −2·75
c) −0·36

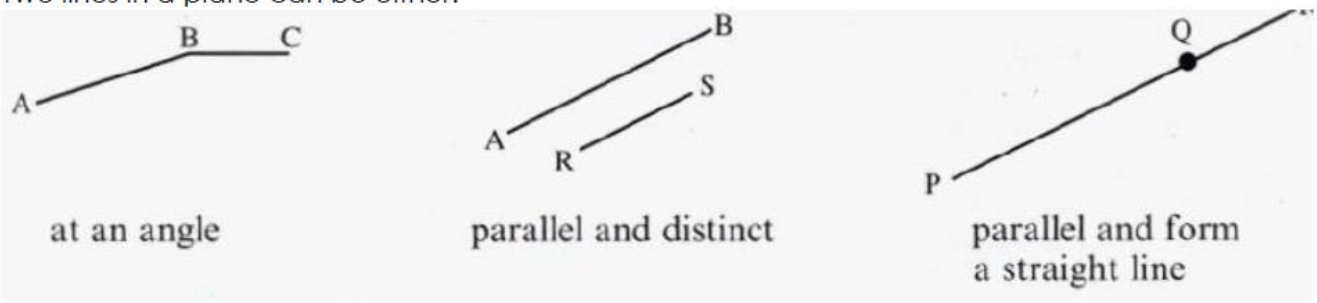
9 −1·6 and 0·4

10 a) $a = 76 \cdot 0°$, $b = 166 \cdot 0°$
b) 90°

11 a) $a = 71 \cdot 6°$, $b = 161 \cdot 6°$
b) The lines are perpendicular.

Tuesday 5th May - Higher Straight Line: Collinearity

Two lines in a plane can be either:



Points that lie on the same straight line are said to be **collinear**.

Show that points A (–6, –2), B (–1, 1) and C (9, 7) are collinear.

SOLUTION

There are three possible line segments to consider, AB, BC and AC. You should find the gradients of any *two* of them.

$$m_{AB} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-2)}{-1-(-6)} = \frac{3}{5} \qquad m_{BC} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{7-1}{9-(-1)} = \frac{6}{10} = \frac{3}{5}$$

Because $m_{AB} = m_{BC}$, AB and BC are parallel.

Since point B is common to both lines, points A, B and C are collinear.

Note

To prove **collinearity**:

- Show that the **gradients** of two line segments are **equal**, which means the lines are parallel.
- State that there is a **common point**.

Later, when you study vectors, you will find another way of proving collinearity.

Exercise

Exercise 1.4

1 Show that points A (1, 3), B (3, 11) and C (5, 19) are collinear. HPQ

2 Show that points D (2, -5), E (4, -11) and F (7, -20) are collinear.

3 Prove that points G (2, 2), H (6, 4) and J (8, 5) all lie on the same straight line.

4 The points (4, 5), (8, 8) and (12, k) are collinear. Find the value of k.

5 The points (1, 10), (2, 13) and (t, 16) all lie on the same straight line. Find t.

6 The points A (3, -2), B (-1, 4) and C (7, q) are collinear. Find q.

7 P is the point (10, -4), Q is (8, 4) and M is the midpoint of PQ.
R is the point (1, -6) and S is (13, 3). Show that R, S and M are collinear.

8 A is the point (3, 7) and B is (9, 19). A third point C lies directly between A and B such that the ratio AC:CB is 1:2. Find the coordinates of C. ACE

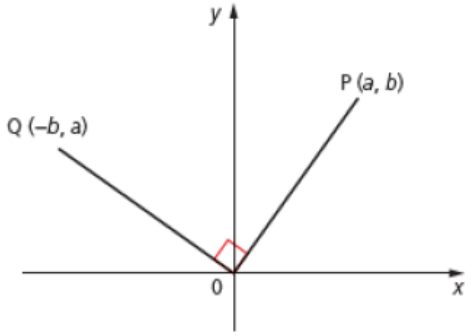
Exercise 1.4

1 Proof
2 Proof
3 Proof
4 $k = 11$
5 $t = 3$
6 $q = -8$
7 Proof
8 (5, 11)

Perpendicular lines

If two lines with gradients m_1 and m_2 are parallel, then $m_1 = m_2$.

In this section we shall consider two lines with gradients m_1 and m_2 which are perpendicular (look back to questions 10 and 11 in Exercise 1.3).



In the above diagram the line OP, where P is the point (a, b), has been rotated 90° anti-clockwise about the origin to OQ, so that Q is the point $(-b, a)$.

Check that $m_{OP} = \frac{b}{a}$ and $m_{OQ} = -\frac{a}{b}$.

Therefore $m_{OP} \times m_{OQ} = \frac{b}{a} \times \left(-\frac{a}{b}\right) = -1$.

This leads to the important result that if two lines with gradients m_1 and m_2 are perpendicular, then $m_1 \times m_2 = -1$.

* To decide if two lines are perpendicular, calculate the product of their gradients. They are perpendicular if the product equals –1.
* If a line has gradient $\frac{e}{f}$ you can find the gradient of a perpendicular line by inverting $\frac{e}{f}$ and changing the sign, leading to $-\frac{f}{e}$. For example, $\frac{5}{2}$ and $-\frac{2}{5}$ are gradients of perpendicular lines.

Examples

1)

If P is the point (2, -3) and Q is the point (-1, 6), find the gradient of a line perpendicular to PQ.

$$M_{PQ} = \frac{6-(-3)}{-1-2} = \frac{9}{-3} = -3$$

Since $\frac{1}{3} \times -3 = -1$, the gradient of a line perpendicular to PQ is $\frac{1}{3}$

2) Triangle RST has coordinates R(1, 2), S(3, 7) and T(6, 0). Show that the triangle is right-angled at R.

$$m_{RS} = \frac{7-2}{3-1} = \frac{5}{2} \qquad m_{RT} = \frac{0-2}{6-1} = -\frac{2}{5}$$

$$m_{RS} \times m_{RT} = \frac{5}{2} \times -\frac{2}{5} = -1$$

Since $m_{RS} \times m_{RT} = -1$, RS is perpendicular to RT and so the triangle is right-angled.

3)

AB is a line perpendicular to the line with equation $3x + 2y = 6$.

Find the equation of AB if A has coordinates $(5, -2)$.

SOLUTION

Rearrange $3x + 2y = 6$ into the form $y = mx + c$.

$$2y = -3x + 6$$

$$\Rightarrow y = -\frac{3}{2}x + 3$$

$$\Rightarrow m = -\frac{3}{2}$$

As AB is perpendicular to the line $3x + 2y = 6$, $m_{AB} = \frac{2}{3}$.

Use the formula $y - b = m(x - a)$ to find the equation of AB.

$$y + 2 = \frac{2}{3}(x - 5)$$

$$\Rightarrow \quad 3y + 6 = 2x - 10$$

$$\Rightarrow 2x - 3y - 16 = 0$$

Exercise 1.5

1 The gradients of six lines are given below:

AB $\frac{1}{2}$ CD $-\frac{2}{5}$ EF -2 GH 3 IJ 3 KL $2{\cdot}5$

State which pairs of lines which are:

a) parallel
b) perpendicular

2 Write down the gradient of a line which is perpendicular to a line with gradient:

a) 4 b) -3 c) $\frac{1}{4}$

d) $-0{\cdot}4$ e) $-\frac{3}{4}$ f) $\frac{7}{2}$

3 Explain why the straight lines with equations $y = 4x + 5$ and $y = -\frac{1}{4}x - 3$ are perpendicular.

4 Write down the equation of the line through the origin perpendicular to the line with equation:

a) $y = 5x$ b) $y = -2x$

c) $y = \frac{1}{2}x$ d) $y = -\frac{1}{6}x$

5 P is the point $(6, 5)$ and Q is $(-3, 4)$

a) Write down the gradient of a line parallel to PQ.
b) Write down the gradient of a line perpendicular to PQ.

6 a) Find the gradient of the line joining points U $(-1, 3)$ and V $(5, -2)$
b) Hence find the equation of the line perpendicular to UV which passes through the point W $(2, 4)$.

7 A is the point $(0, -4)$ and B $(3, 2)$. Find the equation of the line which is **perpendicular** to AB and passes through the point $(5, 1)$. HPQ

8 A line passes through the points G $(1, 3)$ and H $(6, 2)$. Find the equation of the line which is **parallel** to the given line and passes through the point $(-3, 7)$

9 Find the equation of the line passing through C $(-2, 2)$ perpendicular to the line joining A $(-4, 8)$ and B $(1, -3)$.

10 Find the equation of the line passing through L $(4, -1)$ perpendicular to the line joining M $(-3, 0)$ and N $(6, 6)$.

11 G is the point $(-5, -6)$ and H $(-3, -3)$. Find the equation of the line **perpendicular** to GH which passes through the point $(-4, 0)$.

12 Given that K, L and M are the points $(-4, 0)$, $(-1, 3)$ and $(3, -1)$ respectively, prove that triangle KLM is right-angled. ACE

13 A triangle has vertices A $(-3, -3)$, B $(-1, 1)$ and C $(7, -3)$. Show that the triangle is right-angled at B.

14 A line has equation $4x + 3y - 4 = 0$. Find the equation of the line perpendicular to this line which passes through the point $(0, -3)$.

15 AB is a line perpendicular to the line with equation $2x + 4y = 5$.
Find the equation of AB if A has coordinates $(3, 1)$.

16 UV is a line perpendicular to the line with equation $2x + 7y - 1 = 0$.
Find the equation of UV if V has coordinates $(-4, 0)$.

Exercise 1.5

1 a) GH and IJ
b) AB and EF, CD and KL

2 a) $-\frac{1}{4}$

b) $\frac{1}{3}$

c) -4

d) 2·5

e) $\frac{4}{3}$

f) $-\frac{2}{7}$

3 Because $4\times-\frac{1}{4}=-1$

4 a) $y=-\frac{1}{5}x$

b) $y=\frac{1}{2}x$

c) $y=-2x$

d) $y = 6x$

5 a) $\frac{1}{9}$

b) -9

6 a) $-\frac{5}{6}$

b) $6x-5y+8=0$

7 $x+2y-7=0$

8 $x+5y-32=0$

9 $5x-11y+32=0$

10 $3x+2y-10=0$

11 $2x+3y+8=0$

12 Proof

13 Proof

14 $3x-4y-12=0$

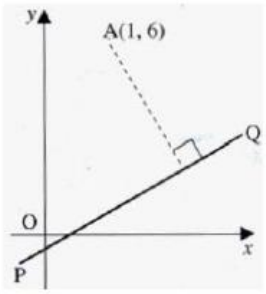
15 $2x-y-5=0$

16 $7x-2y+28=0$

Thursday 7th May - Higher Straight Line: Equations of lines and intersection points

We are now going to put all of this work together and use it, along with our National 5 Simultaneous Equation work to be able to find where lines intersect.

Example: Point A has coordinates (1, 6) and the line PQ has equation $2x - 5y - 1 = 0$.
Find the equation of the line through A perpendicular to PQ
Find the coordinates of the point where this line meets PQ



Eqn of PQ => $5y = 2x - 1$ so $y = \frac{2}{5}x - \frac{1}{5}$ $m = \frac{2}{5}$

$M_{perp} = -\frac{5}{2}$ and using A(1, 6) gives the line

$y - 6 = -\frac{5}{2}(x - 1)$ => $2y - 12 = -5x + 5$ => $5x + 2y - 17 = 0$

Now solving $5x + 2y = 17$ and $2x - 5y = 1$ simultaneously gives:

$10x + 4y = 34$

$10x - 24y = 5$

Subtract $29y = 29$

$y = 1$

Substitute back $5x + 2 = 17$

$5x = 15$

$x = 3$ So intersection point is (3, 1)

EXERCISE 7A

1 Find the point of intersection of the lines given by each pair of equations.

a $3y + x = 7$, $y - x = 1$ **b** $2y + 3x = 10$, $y + x = 5$ **c** $4y + 3x = 1$, $3y - 2x = 5$

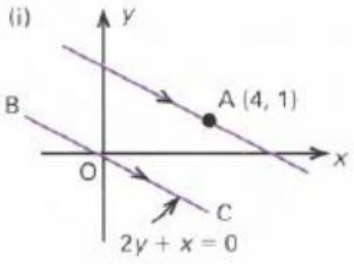
2 Find the coordinates of the points of intersection of the line $2x - 3y = 12$ with the x and y-axes.

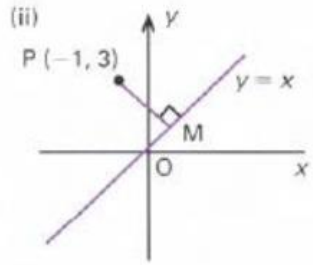
3 The sides of a triangle have equations $x = 1$, $5x + y = 5$ and $2x - y + 5 = 0$. Find the coordinates of the vertices of the triangle.

4 In diagram (i) below, the line through A is parallel to BC, which has equation $2y + x = 0$. Find:

a the equation of the line through A

b the coordinates of the points where it cuts the x and y-axes.





5 In diagram (ii) on page 14, the line through P is perpendicular to the line with equation $y = x$. Find:

a the equation of PM

b the coordinates of M.

6 Parallelogram STUV has vertices S(−3, 3), T(4, 7), U(5, 3) and V(−2, −1). Find:

a the equations of its diagonals

b the coordinates of their point of intersection.

Page 14 Exercise 7A

1 **a** $(1, 2)$ **b** $(0, 5)$ **c** $(-1, 1)$

2 $(6, 0), (0, -4)$

3 $(1, 7), (1, 0), (0, 5)$

4 **a** $2y + x = 6$ **b** $(0, 3), (6, 0)$

5 **a** $y + x = 2$ **b** $(1, 1)$

6 **a** VT, $3y - 4x = 5$; SU, $y = 3$ **b** $(1, 3)$

7 **a** (i) $y - 4x = 4$ (ii) $2y - x = 8$ **b** $(0, 4)$

8 **a** $m_{AB} = \frac{1}{3}$, $m_{BC} = -\frac{1}{2}$

b $y + 3x = 34$, $y - 2x = -16$ **c** $(10, 4)$